

## Expert Opinion in Reliability

by

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GWU/IRRA/Serial TR-92/5  
August 1992



\*Supported by

## The Association of American Railroads

**\*\*Supported by**

Contract N00014-85-K-202  
Office of Naval Research

Grant DAAL03-87-K-0056  
Army Research Office

Grant NRC-G-04-89-104  
Nuclear Regulator Commission

and

Grant AFOSR-F49620-92-J-0030  
Air Force Office of Scientific Research

19950505 117

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## ABSTRACT

In this report we present some results on the formal treatment of expert opinion in reliability analysis. We discuss a procedure which uses opinion of one or two experts for undertaking the reliability assessment of a component. Then we develop the methodology for point processes broadly used in the analysis of defect and count data.

**Keywords:** Weibull Distribution, Nonhomogeneous Poisson Process, Expert Opinion, Life Data, Defect and Failure Counts Data.

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Unannounced <input type="checkbox"/>	
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## 1. INTRODUCTION

The use of expert opinion or informed judgment has become prevalent in practical applications of reliability and risk analysis. This issue is motivated by the fact that modern-day components and systems are designed for high reliability and so the availability of even a small amount of failure data is difficult.

In this paper, we present some results on the formal treatment of expert opinion in reliability analysis. The overall methodology is based on a theme described by Lindley (1983). In Section 2, we introduce a procedure which uses opinion of one or two experts for undertaking the reliability assessment of a component. In Section 3, we discuss the methodology for point processes broadly used in the analysis of defect and failure counts. In Section 4, we present an illustrative real world application. In Section 5, we conclude with some final comments.

## 2. ELICITATION AND MODULATION OF EXPERT OPINION FOR COMPONENT RELIABILITY ANALYSIS USING THE WEIBULL DISTRIBUTION

Consider the unknown quantity  $\mathbf{X}$  = the life length of a unit. Let a Weibull distribution with scale parameter  $\alpha$  and shape parameter  $\beta$  be a meaningful probability model for  $\mathbf{X}$ . Then, the reliability of the unit, given  $\alpha$  and  $\beta$ , is

$$R(x | \alpha, \beta) \equiv \Pr\{\mathbf{X} > x | \alpha, \beta\} = \exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right).$$

Given that  $\alpha$  and  $\beta$  are also unknown quantities, the reliability function of the unit for a given mission time  $x$  is also a random variable. The goal in reliability analysis is to make statements of uncertainty about both  $\mathbf{X}$  and  $R(x | \alpha, \beta)$  conditional on the available information. Note that in our notation the same letter may express either the random variable or a particular realization of such a random variable. Whenever we want to emphasize on a random variable we use the letter in bold. Also, note that we use the terms *unknown quantity* and *random variable* indistinctly.

The Bayesian scheme requires the specification of a prior joint distribution on  $\alpha$  and  $\beta$ , which we denote  $\pi(\alpha, \beta)$ . Suppose that  $\pi(\alpha, \beta)$  is given, then, the reliability of the unit is assessed as

$$R(x) \equiv \Pr\{\mathbf{X} > x\} = \int_{\alpha, \beta} R(x | \alpha, \beta) \pi(\alpha, \beta) d\alpha d\beta. \quad (1)$$

Furthermore, for a fixed mission time  $x$ , a statement of uncertainty about  $R(x | \alpha, \beta)$  can be provided by means of the following expression

$$\Pr\{R(x | \alpha, \beta) \leq c\} = \int_{\{(\alpha, \beta): R(x | \alpha, \beta) \leq c\}} \pi(\alpha, \beta) d\alpha d\beta. \quad (2)$$

Once failure/survival data become available, the prior distribution on  $(\alpha, \beta)$  is updated through Bayes law. Then,  $\pi(\alpha, \beta)$  in Equations (1) and (2) is replaced by the resulting posterior distribution in order to revise the uncertainty about both  $X$  and  $R(x | \alpha, \beta)$ .

We present a procedure for the construction of the prior distribution with the use of expert opinion. The procedure allows the participation of one or two experts as suppliers of expert opinion. An analyst, who is in general the decision maker, monitors the participation of each expert. In the case of two experts, the analyst has to express his (her) opinion about the relationship between the two experts by specifying  $\rho$ , the correlation between the experts. For the Weibull distribution, it is easier to elicit expert opinion on the shape parameter  $\beta$  (characterizing aging), than on the scale parameter  $\alpha$ . Therefore, we introduce the median life  $M$ ,  $M = \alpha (\ln 2)^{1/\beta}$ , and elicit expert opinion on  $M$  and  $\beta$ . Section 2.1 pertains to the elicitation of expert opinion on the median life  $M$ . Section 2.2 pertains to the elicitation of expert opinion on the shape parameter  $\beta$ . Section 2.3 pertains to the assessment of the joint distribution on  $(\alpha, \beta)$  and its revision in the light of observed failure and/or survival data. Whether we have the prior or posterior distribution on  $(\alpha, \beta)$ , Equations (1) and (2) have to be assessed. Advanced numerical techniques can be used to efficiently evaluate the integrals. For details on an implementation of the proposed procedure, refer to Aboura and Campodónico (1992).

Finally, let us point out that a series of assumptions have been made in order to construct the prior distribution, yet other assumptions are of course possible. For example, Singpurwalla and Song (1988) introduced an approach for elicitation of expert opinion on the median life using a chi-square distribution, in which case the reliability assessments were evaluated using approximations from Tierney and Kadane (1986).

## 2.1 Elicitation of expert opinion on the median life $M$ using a Gaussian distribution

Two experts are asked about their opinion on the median life  $M$  of the component. Suppose that expert  $i$  conceptualizes his (her) uncertainty about  $M$  via some distribution with mean  $m_i$  and standard deviation  $s_i$ ,  $i = 1, 2$ .

The analyst views the  $m_i$ 's and the  $s_i$ 's as realizations of random variables and makes some assumptions in order to construct a distribution on  $M$  given the experts' input values. Let " $\propto$ " denote "proportional to," then the density of interest is

$$p(M | m_1, m_2, s_1, s_2) \propto p(m_1, m_2, s_1, s_2 | M) p(M) = p(m_1, m_2 | s_1, s_2, M) p(s_1, s_2 | M) p(M).$$

The analyst assumptions are

Assumption 1:  $p(M)$  is effectively constant,

Assumption 2:  $p(s_1, s_2 | M)$  does not depend on  $M$ ,

Assumption 3:  $p(m_1, m_2 | s_1, s_2, M)$  is a bivariate normal distribution with  $E(m_i) = b_i M + a_i$ ,  $\text{Var}(m_i) = (\gamma_i s_i)^2$ ,  $i = 1, 2$ , and correlation  $\rho$  between  $m_1$  and  $m_2$ .

The parameters  $a_i$ ,  $b_i$  and  $\gamma_i$  reflect the analyst judgment of the expertise of expert  $i$ ,  $i = 1, 2$ , and  $\rho$  is the correlation between the two experts in the analyst opinion. For example, if the analyst considers that expert  $i$  is unbiased in the specification of  $m_i$ , then  $a_i = 0$  and  $b_i = 1$ ; if  $b_i = 1$ , then  $a_i$  is a bias term that expresses the analyst view that expert  $i$  overestimates (underestimates)  $M$  by an amount  $a_i$ ; if the analyst thinks that expert  $i$  overestimates (underestimates)  $M$  by some percentage, say 10%, then  $a_i = 0$ ,  $b_i = 1.1$  (0.9); finally, if the analyst thinks that expert  $i$  exaggerates (is cautious about) the standard deviation  $s_i$ , then  $\gamma_i > (<) 1$ .

Under the above three assumptions,  $p(M | m_1, m_2, s_1, s_2) \propto p(m_1, m_2 | s_1, s_2, M)$  from which the following results [see Lindley (1983)]: The uncertainty about  $M$  is described via a normal distribution with mean

$$\mu = \frac{\sum_{i,j=1}^2 b_i \sigma^{ij} (m_j - a_j)}{\sum_{i,j=1}^2 b_i \sigma^{ij} b_j},$$

and standard deviation

$$\sigma = \left( \sum_{i,j=1}^2 b_i \sigma^{ij} b_j \right)^{-1/2},$$

where  $\sigma^{ij}$  are elements of the matrix that is the inverse of that with elements  $\sigma_{ii} = (\gamma_i s_i)^2$ ,  $i = 1, 2$ ,

and  $\sigma_{12} = \rho(\sigma_{11}\sigma_{22})^{1/2} = \sigma_{21}$ . Since  $M$  is nonnegative, we consider the truncated density of  $M$  as

$$\pi(M | \mu, \sigma) = \frac{K}{\sqrt{2\pi}\sigma} \exp\left(-0.5\left(\frac{M-\mu}{\sigma}\right)^2\right), \quad M > 0,$$

where

$$K = \left( \int_{-\mu/\sigma}^{\infty} (2\pi)^{-1/2} \exp\left(-\frac{u^2}{2}\right) du \right)^{-1}.$$

## 2.2 Elicitation of expert opinion on the shape parameter $\beta$

The procedure assumes that expert  $i$ ,  $i = 1, 2$ , conceptualizes the uncertainty about  $\beta$ , the shape parameter of the Weibull distribution, via a lognormal distribution with density

$$\pi(\beta | \lambda_i, p_i) = \frac{1}{\sqrt{2\pi} p_i \beta} \exp\left(-\frac{1}{2p_i^2}(\ln\beta - \lambda_i)^2\right).$$

The parameters  $\lambda_i$  and  $p_i$  are called the "log mean" and the "log standard deviation", respectively. The mean and the standard deviation of the lognormal distribution are

$$e^{\lambda_i + p_i^2/2} \quad \text{and} \quad \sqrt{e^{2\lambda_i + p_i^2} (e^{p_i^2} - 1)},$$

respectively. The values of  $\lambda_i$  and  $p_i$  are pinned down by questioning expert  $i$  about the rate of deterioration of the component. Having assigned values to  $\lambda_i$  and  $p_i$ ,  $i = 1, 2$ , and under assumptions similar to Assumptions 1 to 3 in section 2.1, the analyst describes the uncertainty about  $\beta$  via a lognormal distribution with log mean parameter

$$\mu' = \frac{\sum_{i,j=1}^2 \xi^{ij} \lambda_i}{\sum_{i,j=1}^2 \xi^{ij}},$$

and log standard deviation parameter

$$\sigma' = \left( \sum_{i,j=1}^2 \xi^{ij} \right)^{-1/2},$$

where  $\xi^{ij}$  are elements of the matrix that is the inverse of that with elements  $\xi_{ii} = p_i^2$ ,  $i = 1, 2$ , and  $\xi_{12} = \xi_{21} = \rho(\xi_{11}\xi_{22})^{1/2}$ .

In other words, the analyst ends up with the following prior distribution on  $\beta$

$$\pi(\beta | \mu', \sigma') = \frac{1}{\sqrt{2\pi} \sigma' \beta} \exp\left(-\frac{1}{2\sigma'^2}(\ln\beta - \mu')^2\right).$$

### 2.3 Prior and posterior joint distribution on $(\alpha, \beta)$

Since  $M = \alpha(\ln(2))^{1/\beta}$ , then the prior density of  $\alpha$  is given by

$$\pi(\alpha | \beta, \mu, \sigma) = \frac{K(\ln(2))^{1/\beta}}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2}\left(\frac{\alpha(\ln(2))^{1/\beta} - \mu}{\sigma}\right)^2\right), \quad \alpha > 0;$$

and the prior joint density of  $\alpha$  and  $\beta$  is

$$\begin{aligned} \pi(\alpha, \beta | \mu, \sigma, \mu', \sigma') &= \pi(\alpha | \beta, \mu, \sigma) \pi(\beta | \mu', \sigma') \\ &= \frac{K(\ln(2))^{1/\beta}}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2}\left(\frac{\alpha(\ln(2))^{1/\beta} - \mu}{\sigma}\right)^2\right) \frac{1}{\sqrt{2\pi} \sigma' \beta} \exp\left(-\frac{1}{2\sigma'^2}(\ln\beta - \mu')^2\right). \end{aligned}$$

Once life data are obtained, say  $D = (x_1, \dots, x_n, s_1, \dots, s_m)$  where the  $x_i(s_i)$ 's are failure (survival) data, the *likelihood* of  $\alpha$  and  $\beta$  is obtained as

$$\mathcal{L}(\alpha, \beta | D) = \prod_{i=1}^n \left(\frac{\beta}{\alpha}\right) \left(\frac{x_i}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{x_i}{\alpha}\right)^\beta\right) \prod_{j=1}^m \exp\left(-\left(\frac{s_j}{\alpha}\right)^\beta\right);$$

i.e., the likelihood is the joint density of the observed values when regarded as a function of the parameters. Then, the posterior joint density of  $(\alpha, \beta)$  can be evaluated through Bayes law,

$$\pi(\alpha, \beta | D, \mu, \sigma, \mu', \sigma') \propto \mathcal{L}(\alpha, \beta | D) \pi(\alpha, \beta | \mu, \sigma, \mu', \sigma').$$

### 3. ELICITATION AND MODULATION OF EXPERT OPINION FOR A POINT PROCESS

Consider now the stochastic process  $\{N(t), t \geq 0\}$  where  $N(t)$  represents the total counts which occur up to time  $t$ . A counting process finds multiple applications in modeling the number of defects or failures in a system. Suppose that  $\{N(t), t \geq 0\}$  is modeled as a non-homogeneous Poisson process (NHPP) with *intensity function*  $r(t | \Theta)$  and *mean value function*  $M(t | \Theta) \equiv E(N(t) | \Theta)$ , where  $\Theta$  is a

vector of model parameters. In a NHPP the mean value function and the intensity function are related as follows.

$$M(t | \Theta) = \int_0^t r(s | \Theta) ds.$$

Conversely, the intensity function represents the instantaneous rate of change of the expected number of counts, i.e.,

$$\frac{d}{dt}M(t | \Theta) = r(t | \Theta).$$

Therefore, if  $r(t | \Theta)$  is an increasing function on  $t$ , then counts are expected to be more frequent as time increases. If  $r(t | \Theta)$  is a decreasing function on  $t$ , then counts are expected to be more scarce as time evolves.

Suppose that the uncertainty on  $\Theta$  is represented by the joint distribution  $\pi(\Theta)$ , then the aim of a NHPP analysis is to estimate quantities of interest such as

- The expected number of counts in the interval  $(0, t]$  :

$$E(N(t)) = M(t) = \int_{\Theta} M(t | \Theta) \pi(\Theta) d\Theta. \quad (3)$$

- The expected number of counts in the interval  $(s, t]$ ,  $s < t$  :

$$E(N(s, t)) = \int_{\Theta} (M(t | \Theta) - M(s | \Theta)) \pi(\Theta) d\Theta. \quad (4)$$

- The probability of  $k$  counts in the interval  $(0, t]$ ,  $k = 0, 1, 2, \dots$  :

$$\Pr\{N(t) = k\} = \int_{\Theta} \frac{e^{-M(t | \Theta)} [M(t | \Theta)]^k}{k!} \pi(\Theta) d\Theta. \quad (5)$$

- The probability of  $k$  counts in the interval  $(s, t]$ ,  $s < t$ ,  $k = 0, 1, 2, \dots$  :

$$\Pr\{N(s, t) = k\} = \int_{\Theta} \frac{e^{-(M(t | \Theta) - M(s | \Theta))} [M(t | \Theta) - M(s | \Theta)]^k}{k!} \pi(\Theta) d\Theta. \quad (6)$$



Our goal now is to continue with the general scheme presented in the previous sections and use expert opinion in the construction of a prior distribution on the model parameters (or a transformation of the parameters). In the following we consider two well-used processes and develop a series of assumptions for the elicitation and modulation of expert opinion for their analyses. We consider one expert only, and as before, expert opinion is monitored by an analyst.

The first process under consideration is the power-law process, also known as the Weibull process because of the facts that its intensity function behaves like the failure rate function of a Weibull distribution and the occurrence time of the first count is Weibull distributed. The Weibull process is quite flexible in the sense that its intensity function can be either increasing, constant or decreasing. The second process considered here is known as the Logarithmic Poisson model which has become one of the more popular models in software reliability. The intensity function for the Logarithmic Poisson process is always a decreasing function and its derivation encapsulates the belief that the intensity function decreases exponentially with the number of failures experienced [see Musa and Okumoto (1984)]. Section 3.1 pertains to the elicitation and modulation of expert opinion in the Weibull process. Section 3.2 accommodates the same approach to the Logarithmic Poisson process.

### 3.1 Elicitation and modulation of expert opinion for the Weibull process

The intensity function and the mean value function of the Weibull process are

$$r(t | \alpha, \beta) = \frac{\beta}{\alpha} \left( \frac{t}{\alpha} \right)^{\beta-1} \text{ and } M(t | \alpha, \beta) = \left( \frac{t}{\alpha} \right)^{\beta}, t \geq 0,$$

respectively, where  $\alpha$  is the scale parameter and  $\beta$  is the shape parameter. The model is reparametrized by introducing two unknown quantities for which the expert visualizes a prior distribution more naturally. Consider two index points  $T_1$  and  $T_2$ , such that  $T_1 < T_2$ , and define the unknown quantities  $M_1$  and  $M_2$  as

$$M_1 = (T_1/\alpha)^{\beta} \text{ and } M_2 = (T_2/\alpha)^{\beta}. \quad (7)$$

That is,  $M_i$  represents the expected number of counts in the interval  $[0, T_i]$ , for  $i = 1, 2$ . Given  $T_1$  and  $T_2$ , Equation (7) provides a relationship between  $(M_1, M_2)$  and  $(\alpha, \beta)$  from which  $\alpha$  and  $\beta$  can be explicitly determined in terms of  $M_1$  and  $M_2$ . Let  $\alpha = \alpha(M_1, M_2)$  and  $\beta = \beta(M_1, M_2)$ , where  $\alpha(\cdot, \cdot)$  and  $\beta(\cdot, \cdot)$  are the corresponding transformation functions. Now, the model can be written in terms

of  $(\mathbf{M}_1, \mathbf{M}_2)$ . For example, the mean value function of the process conditioned on  $\mathbf{M}_1$  and  $\mathbf{M}_2$ , denoted by  $M(t | \cdot)$ , takes the following equivalent form

$$M(t | \cdot) \equiv M(t | \mathbf{M}_1, \mathbf{M}_2) = \left( \frac{t}{\alpha(\mathbf{M}_1, \mathbf{M}_2)} \right)^{\beta(\mathbf{M}_1, \mathbf{M}_2)}.$$

A prior distribution on  $(\mathbf{M}_1, \mathbf{M}_2)$ , say  $\pi(\mathbf{M}_1, \mathbf{M}_2)$ , is obtained from the interaction of an expert and an analyst. For each  $\mathbf{M}_i$ , the expert  $\mathcal{E}$  visualizes a distribution, say  $p_{\mathbf{M}_i}^{\mathcal{E}}(\cdot)$ ,  $i = 1, 2$ , and declares to the analyst  $\mathcal{A}$  two numbers  $m_i$  and  $s_i$  as measures of the location and the scale of  $p_{\mathbf{M}_i}^{\mathcal{E}}(\cdot)$ , respectively. The analyst  $\mathcal{A}$  views the  $m_i$ 's and the  $s_i$ 's as realizations of two random variables  $\mathcal{M}_i$  and  $\mathcal{J}_i$ ,  $i = 1, 2$ , and models the distributions of  $\mathcal{M}_i$  and  $\mathcal{J}_i$  based on both his (her) opinion ( $\mathcal{A}$ 's opinion) of the expertise of  $\mathcal{E}$  and his (her) perceived correlation between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  and  $\mathcal{J}_1$  and  $\mathcal{J}_2$ . Table 1 summarizes our notation.

**Table 1. Summary of the Notation**

Nature		Mean	St.Deviation
<b><math>\mathbf{M}_1</math></b> Expected counts in $(0, T_1]$	Random quantity	$\mathcal{M}_1$	$\mathcal{J}_1$
	Expert's declared value	$m_1$	$s_1$
<b><math>\mathbf{M}_2</math></b> Expected counts in $(0, T_2]$	Random quantity	$\mathcal{M}_2$	$\mathcal{J}_2$
	Expert's declared value	$m_2$	$s_2$

The goal is to use the  $m_i$ 's and the  $s_i$ 's,  $i = 1, 2$ , and  $\mathcal{A}$ 's view of the expertise of the expert to arrive at a joint distribution for  $\mathbf{M}_1$  and  $\mathbf{M}_2$ .

Since  $\mathcal{E}$  assesses both  $\mathbf{M}_1$  and  $\mathbf{M}_2$  and since  $\mathbf{M}_1 \leq \mathbf{M}_2$ , it is reasonable to suppose that  $\mathcal{M}_1$  and  $\mathcal{M}_2$  will be dependent and positively so. That is, large values of  $\mathcal{M}_1$  will lead to large values of  $\mathcal{M}_2$  and vice versa. Also, the declared values  $m_1$  and  $m_2$  are such that  $m_1 < m_2$ .

Given  $m_1, m_2, s_1$  and  $s_2$ , we need to obtain  $p(\mathbf{M}_1, \mathbf{M}_2 | m_1, m_2, s_1, s_2)$ . By Bayes law,

$$p(\mathbf{M}_1, \mathbf{M}_2 | m_1, m_2, s_1, s_2) \propto p(m_1, m_2, s_1, s_2 | \mathbf{M}_1, \mathbf{M}_2) p(\mathbf{M}_1, \mathbf{M}_2).$$

The first term on the right hand side is the likelihood and can be factored as

$$p(m_1, m_2, s_1, s_2 | M_1, M_2) =$$

$$p_{\mathcal{M}_2}(m_2 | m_1, s_1, s_2, M_1, M_2) p_{\mathcal{Y}_2}(s_2 | m_1, s_1, M_1, M_2) p_{\mathcal{M}_1}(m_1 | s_1, M_1, M_2) p_{\mathcal{Y}_1}(s_1 | M_1, M_2).$$

The subscripts associated with each  $p$  pertain to the random variable upon whose distribution  $\mathcal{A}$ 's likelihood is based. To evaluate the above likelihood,  $\mathcal{A}$  makes a series of assumptions. A justification for assumptions of this type has been given by Lindley (1983). The assumptions are:

A1.  $\mathcal{M}_2$  is independent of  $M_1$  given  $\mathcal{M}_1$ ,  $M_2$ ,  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$ ;

$$\text{thus } p_{\mathcal{M}_2}(m_2 | m_1, s_1, s_2, M_1, M_2) = p_{\mathcal{M}_2}(m_2 | m_1, s_1, s_2, M_2).$$

A2.  $p_{\mathcal{M}_2}(m_2 | m_1, s_1, s_2, M_2)$  is a truncated normal with a mean  $a + bM_2$  and standard deviation  $\gamma s_2$ ; the truncation is on the left and the point of truncation is  $m_1 + ks_1$ , where  $k$  is specified by  $\mathcal{A}$ . The constants  $a$ ,  $b$  and  $\gamma$  reflect  $\mathcal{A}$ 's views on the expertise of  $\mathcal{S}$ . For an interpretation of  $a$ ,  $b$  and  $\gamma$ , see Lindley (1983).

A3.  $\mathcal{Y}_2$  is independent of  $\mathcal{M}_1$  given  $\mathcal{Y}_1$ ,  $M_1$  and  $M_2$ ; thus  $p_{\mathcal{Y}_2}(s_2 | m_1, s_1, M_1, M_2) = p(s_2 | s_1, M_1, M_2)$ .

A4.  $p_{\mathcal{Y}_2}(s_2 | s_1, M_1, M_2)$  is a truncated normal with mean  $s_1$  and variance  $(M_2 - M_1)$ ; the truncation is on the left and the point of truncation is 0.

A5.  $\mathcal{M}_1$  is independent of  $M_2$  given  $M_1$  and  $\mathcal{Y}_1$ ; thus  $p_{\mathcal{M}_1}(m_1 | s_1, M_1, M_2) = p_{\mathcal{M}_1}(m_1 | s_1, M_1)$ .

A6.  $p_{\mathcal{M}_1}(m_1 | s_1, M_1)$  is a truncated normal with mean  $a + bM_1$  and standard deviation  $\gamma s_1$ . The point of truncation is on the left and is at 0. The parameters  $a$ ,  $b$  and  $\gamma$  are the same as in A2.

A7.  $p_{\mathcal{Y}_1}(s_1 | M_1, M_2)$  is exponential with mean  $(M_2 - M_1)$ . This implies that as the disparity between  $M_1$  and  $M_2$  increases, the uncertainty  $\mathcal{Y}_1$  becomes larger and larger.

A8.  $p(M_1, M_2)$  is relatively constant over the range of  $M_1$  and  $M_2$  on which the likelihood is appreciable.

**Theorem 1.**

Under Assumptions A1-A8, the joint distribution of  $\mathbf{M}_1$  and  $\mathbf{M}_2$ , given  $m_1, m_2, s_1, s_2$  and  $\mathcal{A}$ 's assessment of the expertise of  $\mathcal{S}$  (denoted by  $\mathfrak{S}$ ) is given by

$$p(\mathbf{M}_1, \mathbf{M}_2 \mid m_1, m_2, s_1, s_2, \mathfrak{S}) \propto$$

$$\frac{\exp\left(-\frac{1}{2}\left(\frac{m_2 - a - b\mathbf{M}_2}{\gamma s_2}\right)^2\right)}{\left(1 - \Phi\left(\frac{m_1 + ks_1 - a - b\mathbf{M}_2}{\gamma s_2}\right)\right) \gamma s_2} \times \frac{\exp\left(-\frac{1}{2}\left(\frac{s_2 - s_1}{\mathbf{M}_2 - \mathbf{M}_1}\right)^2\right)}{\left(1 - \Phi\left(\frac{-s_1}{\mathbf{M}_2 - \mathbf{M}_1}\right)\right) (\mathbf{M}_2 - \mathbf{M}_1)} \times \frac{\exp\left(-\frac{1}{2}\left(\frac{m_1 - a - b\mathbf{M}_1}{\gamma s_1}\right)^2\right)}{\left(1 - \Phi\left(\frac{-a - b\mathbf{M}_1}{\gamma s_1}\right)\right) \gamma s_1} \times \frac{\exp\left(\frac{-s_1}{\mathbf{M}_2 - \mathbf{M}_1}\right)}{\mathbf{M}_2 - \mathbf{M}_1}$$

where  $0 \leq \mathbf{M}_1 \leq \mathbf{M}_2 < \infty$ , and  $\Phi(u) = \int_{-\infty}^u \frac{\exp(-x^2/2)}{\sqrt{2\pi}} dx$ .

Suppose now that the number of counts in a total of  $N$  disjoint intervals are observed. Let  $\eta_j$  = observed number of counts in the interval  $(t_{1j}, t_{2j}]$ ,  $j = 1, \dots, N$ . Let

$$D = \{\eta_j, t_{1j}, t_{2j} : j = 1, \dots, N\}$$

denote the data set. We update the distribution on  $(\mathbf{M}_1, \mathbf{M}_2)$  by means of Bayes law which establishes that

$$\pi(\mathbf{M}_1, \mathbf{M}_2 \mid D) \propto \mathcal{L}(\mathbf{M}_1, \mathbf{M}_2 \mid D) \pi(\mathbf{M}_1, \mathbf{M}_2). \quad (8)$$

The likelihood function  $\mathcal{L}(\mathbf{M}_1, \mathbf{M}_2 \mid D)$ , the joint density of the observed values when regarded as a function of the parameters, is given by

$$\mathcal{L}(\mathbf{M}_1, \mathbf{M}_2 \mid D) = \prod_{j=1}^n \frac{e^{-(M(t_{2j} \mid \cdot) - M(t_{1j} \mid \cdot))} (M(t_{2j} \mid \cdot) - M(t_{1j} \mid \cdot))^{\eta_j}}{\eta_j!}.$$

The constant of proportionality for Equation (8) is such that  $\pi(\mathbf{M}_1, \mathbf{M}_2 \mid D)$  obeys the rules of probability, i.e., it integrates to one over the range of  $(\mathbf{M}_1, \mathbf{M}_2)$ .

### 3.2 Elicitation and modulation of expert opinion for the Logarithmic Poisson process

An equivalent procedure can be developed for the Logarithmic Poisson process whose intensity function and mean value function are given by

$$r(t | \theta, \lambda) = \frac{\lambda}{\lambda \theta t + 1} \text{ and } M(t | \theta, \lambda) = \frac{1}{\theta} \ln(\lambda \theta t + 1), t \geq 0,$$

respectively, where  $\theta, \lambda > 0$  are the model parameters, two unknown quantities. We reparametrize the model as in section 3.1. Given  $T_1$  and  $T_2$  such that  $T_1 < T_2$ , we define the unknown quantity  $M_i = M(T_i | \theta, \lambda) = \frac{1}{\theta} \ln(\lambda \theta T_i + 1)$ ,  $i = 1, 2$ . The system of equations

$$\begin{cases} M_1 = \ln(\lambda \theta T_1 + 1)/\theta \\ M_2 = \ln(\lambda \theta T_2 + 1)/\theta \end{cases}$$

has a solution on  $(\theta, \lambda)$ ,  $\theta > 0$ , if and only if  $M_1 < M_2 < \frac{T_2}{T_1} M_1$ . This fact should be taken into account when eliciting a joint distribution on  $(M_1, M_2)$ .

As before,  $\mathcal{S}$  visualizes a distribution for each  $M_i$ , say  $p_{\mathcal{S}}^{M_i}(\cdot)$ ,  $i = 1, 2$ , and declares to the analyst  $\mathcal{A}$  two numbers  $m_i$  and  $s_i$  as measures of the location and the scale of  $p_{\mathcal{S}}^{M_i}(\cdot)$ , respectively. The declared values  $m_1$  and  $m_2$  are such that  $m_1 < m_2 < (T_2/T_1) m_1$ .

The analyst  $\mathcal{A}$  views the  $m_i$ 's and the  $s_i$ 's as realizations of two random variables  $\mathcal{M}_i$  and  $\mathcal{J}_i$ ,  $i = 1, 2$ , and models the distributions of  $\mathcal{M}_i$  and  $\mathcal{J}_i$  based on both his (her) opinion of the expertise of  $\mathcal{S}$  and his (her) perceived correlation between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  and  $\mathcal{J}_1$  and  $\mathcal{J}_2$ .

We consider the same assumptions as in section 3.1 with a slight modification in Assumption A2 in order to satisfy the requirement that  $M_1 < M_2 < (T_2/T_1) M_1$ . Let us rewrite the corresponding Assumption A2'.

A2'.  $p_{\mathcal{M}_2}(m_2 | m_1, s_1, s_2, M_2)$  is a truncated normal with a mean  $a + bM_2$  and standard deviation  $\gamma s_2$ ; the truncation is on the left and right. The left point of truncation is  $m_1 + ks_1$ , where  $k$  is specified by  $\mathcal{A}$ . The right point of truncation is  $(T_2/T_1) m_1$ . The constants  $a$ ,  $b$  and  $\gamma$  reflect  $\mathcal{A}$ 's views on the expertise of the expert. For an interpretation of  $a$ ,  $b$  and  $\gamma$ , see Lindley (1983).

Theorem 2. Under Assumptions A1-A8 (from Section 3.1), with A2 replaced by A2', the joint distribution of  $M_1$  and  $M_2$ , given  $m_1, m_2, s_1, s_2$  and  $\mathcal{A}$ 's assessment of the expertise of  $\mathcal{S}$  (denoted by  $\mathfrak{E}$ ) is given by

$$p(M_1, M_2 | m_1, m_2, s_1, s_2, \mathfrak{E}) \propto$$

$$\frac{\exp\left(-\frac{1}{2}\left(\frac{m_2 - a - bM_2}{\gamma s_2}\right)^2\right)}{\left(\Phi\left(\frac{\frac{T_2}{T_1}m_1 - a - bM_2}{\gamma s_2}\right) - \Phi\left(\frac{m_1 + ks_1 - a - bM_2}{\gamma s_2}\right)\right) \gamma s_2} \times \frac{\exp\left(-\frac{1}{2}\left(\frac{s_2 - s_1}{M_2 - M_1}\right)^2\right)}{\left(1 - \Phi\left(\frac{-s_1}{M_2 - M_1}\right)\right) (M_2 - M_1)} \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\left(\frac{m_1 - a - bM_1}{\gamma s_1}\right)^2\right)}{\left(1 - \Phi\left(\frac{-a - bM_1}{\gamma s_1}\right)\right) \gamma s_1} \times \frac{\exp\left(\frac{-s_1}{M_2 - M_1}\right)}{M_2 - M_1},$$

where  $0 < M_1 < M_2 < \frac{T_2}{T_1} M_1$ , and  $\Phi(u) = \int_{-\infty}^u \frac{\exp(-x^2/2)}{\sqrt{2\pi}} dx$ .

This expression provides the prior joint density distribution on  $M_1$  and  $M_2$ . As in Section 3.1, this distribution can be updated in the light of observed data.

#### 4. APPLICATION

In this Section, we present an illustrative scenario from the railroad industry using a Weibull process (Section 3.1).

A railroad track ages and develops fatigue defects up to a point where they become excessive and the track, or part of it, has to be replaced. The optimal schedule for the replacement of the track depends on, among other variables, the number of fatigue defects predicted for a future time. A Weibull process, indexed over load traffic (in million gross tons), is used to model the number of fatigue defects on the track segment. In order to incorporate the length of the segment, say  $L$  miles, we slightly modify the intensity function as

$$r(t | \alpha, \beta) = L \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1}.$$

Expert opinion on the track segment is generally based on the outcome of engineering/deterministic models. In this form, the physics of the rail deterioration is incorporated into the model as prior information. Thereafter, the natural inclusion of observed defects into the model provides further possibilities of better predictions, overcoming a limitation of engineering/deterministic models which can not be automatically calibrated as data become available. A customized computer implementation of the Weibull process using expert opinion has been developed for the Association of American Railroads [see Campodónico (1992)].

In order to demonstrate the evolution of the prediction process we present some results for a railroad segment of length  $L = 105$  miles, which we call Railroad A. Based on the outcomes of an engineering model, the railroad expert selects two million-gross-ton (MGT) intervals, both starting at zero. He/she then enters the most likely number of defects in each interval and a subjective measure of uncertainty or variability. Figure 1 reproduces a computer screen for the input of expert opinion.

Defect data on Railroad A was collected starting in 1986, by which time the track had accumulated 600 MGT. The annual traffic for that year was 45 MGT, and the number of observed fatigue defects was 16. We also obtained data on the annual traffic and observed defects for the years 1987-1990. In order to illustrate the evolution of the prediction process, let us suppose that we start at the end of year 1985 and that the traffic (in MGT) for the next five years (1986 to 1990) is known. In 1985, the number of defects for the future annual traffic intervals are estimated based on expert opinion alone. By the end of 1986, the first data become available so that we update the predictions for the following 4 years (1987 to 1990). The updating procedure can be repeated by the end of each year once the number of defects observed during the ending year is released. Figure 2 summarizes the above exercise by showing the expected number of defects predicted per annual MGT interval. The predictions for each interval can then be compared to the actual number of observed defects. In addition to the expected values, a probability distribution for the number of defects in any MGT interval can also be calculated. Figure 3, for example, reproduces the prior and posterior probability distributions of the number of fatigue defects in the interval (818, 885] MGT. The posterior distribution corresponds to the combination of the prior information and the observations from years 1986 to 1989. We observe that the posterior distribution is not only shifted to the right, but also sharper, i.e., the uncertainty has been reduced.

# INPUT EXPERT OPINION

Enter the following information: (I/O drive: C)

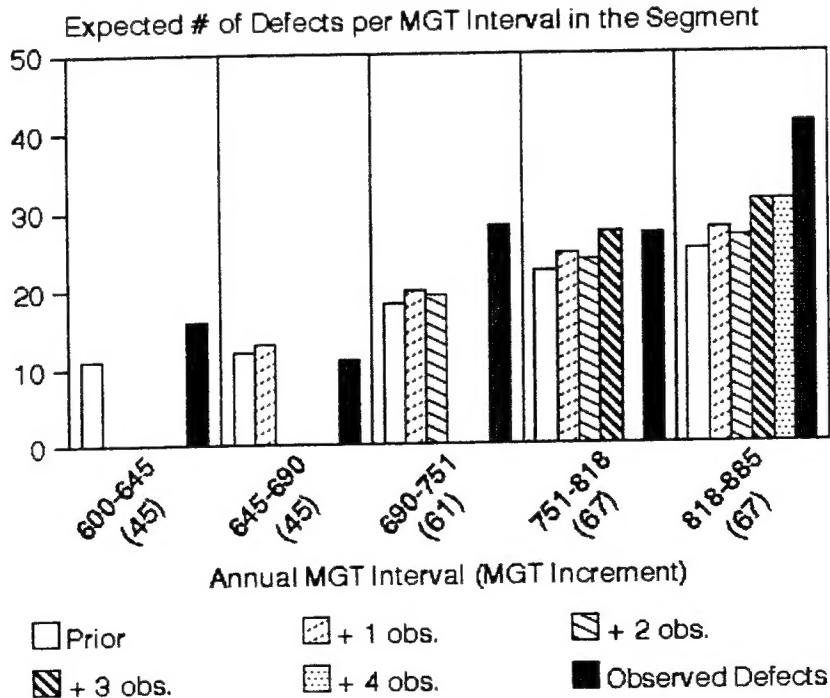
- Two MGT intervals (both starting from zero) within which the number of defects per mile can be best predicted.

First interval: (0,

- Your best guess about the number of defects per mile in the FIRST interval,  $m1 =$  .
- A 99% credibility interval around  $m1$  ( 0.6000), i.e., you are almost sure that the true value for the number of defects per mile in the FIRST interval will lie between  and  defects.
- Your best guess about the number of defects per mile in the SECOND interval,  $m2 =$  .
- A 99% credibility interval around  $m1$  ( 1.8000), i.e., you are almost sure that the true value for the number of defects per mile in the SECOND interval will lie between  and  defects.

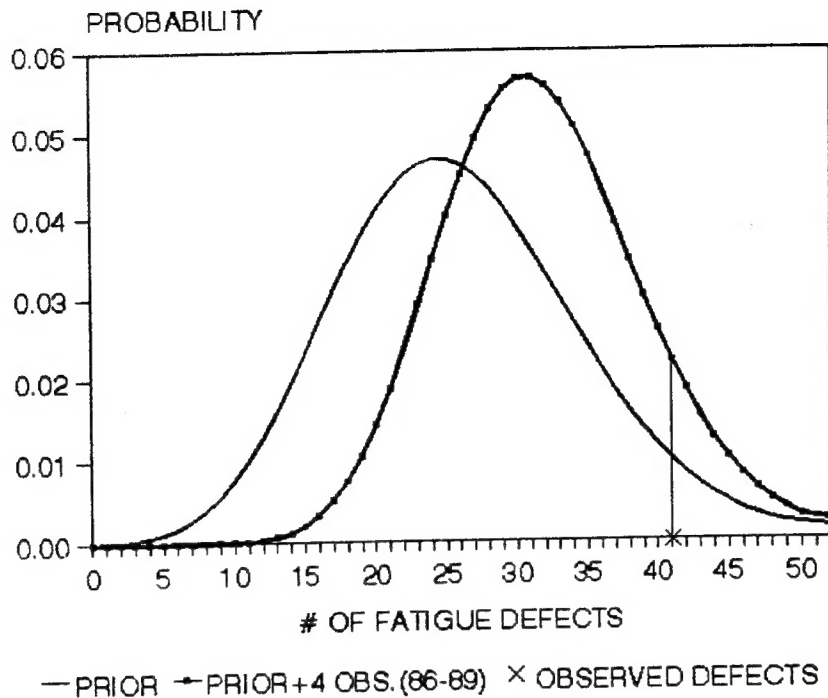
- An identification message concerning the above input (optional):

**Figure 1.** Screen for the input of expert opinion.



**Figure 2.** Expected defects as they evolve with additional data.





**Figure 3.** Probability Distribution for the number of Defects in the traffic interval (818, 885] MGT (corresponding to 1990).

## 5. FINAL COMMENTS

Computer implementations of the procedures presented in this article are required in order to evaluate the integrals involved in the assessment of the quantities of interest given by Equations (1) and (2) for the reliability assessment of a component, and by Equations (4) and (6) for estimating the number of counts or defects in future time intervals. Some examples of general numerical techniques used for the evaluation of integrals are the Gauss-Laguerre approximations, the Tierney-Kadane approximations, and simulation techniques such as Monte-Carlo integration and, more recently, the Gibbs sampling. The rapid advances in computer technology as well as in numerical methods promote the choice of model assumptions which are intuitively reasonable rather than merely mathematically convenient. The approach here has been to look at expert opinion as data and to construct their likelihood.

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